Convolution via exponentiation

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Background

Convolution for monoidal categories

Given a small monoidal category $(\mathcal{C}, \otimes, I)$, the category of presheaves $\widehat{\mathcal{C}} := [\mathcal{C}^{op}, \mathrm{Set}]$ inherits a monoidal structure [Day 1970].

The tensor product is defined via a convolution formula:

$$p \mathbin{\hat{\otimes}} q := c \mapsto \int^{x,y \in \mathcal{C}} p(x) \times q(y) \times \mathcal{C}(c, x \otimes y)$$

The unit is given by the representable $\mathcal{C}(-, I)$.

The convolution of convolution

Establishing that this indeed defines a monoidal structure is an exercise in the coend calculus. While not particularly difficult for those versed in the necessary manipulations, one can't help but think there must be a simpler way.

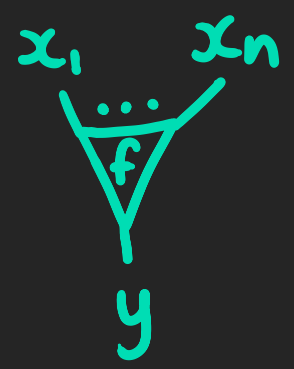
The key is to spot that the tensor product in $\hat{\mathcal{C}}$ is defined by a colimit, which has a mapping out universal property.

This indicates it may be helpful to consider the construction multicategorically.

Multicategories

A multicategory is like a category, but where morphisms have multiary domain.

Every (colax) monoidal category can be viewed as a multicategory, in which a multiary morphism is given by a morphism from a tensor product.



$$:= \qquad \begin{array}{c} x_1 \otimes \cdots \otimes x_n \\ \Big\downarrow f \\ y \end{array}$$

These are the (weakly) representable multicategories.

Convolution, multicategorically

Natural transformations in $\widehat{\mathscr{C}}$

$$\int^{x,y\in\mathscr{C}} p(x)\times q(y)\times \mathscr{C}(-,x\otimes y) \Rightarrow r$$

are in bijection with families of functions

$$p(x)\times q(y)\longrightarrow r(x\otimes y)$$

natural in $x,y\in\mathscr{C}$. Similarly, natural transformations

$$\mathscr{C}(-,I)\Rightarrow r$$

are in bijection with elements of $r(I)$.

Convolution, multicategorically

$\widehat{\mathscr{C}}$ is equipped with the structure of a multicategory, for which a multimorphism $p_1, \dots, p_n \longrightarrow r$ is a family

$$p_1(x_1) \times p_2(x_2) \times \cdots \times p_n(x_n) \longrightarrow r(x_1 \otimes x_2 \otimes \cdots \otimes x_n)$$

natural in each $x_i \in \mathscr{C}$.

It is very easy to show this forms a multicategory; that this multicategory is representable (hence arises from a monoidal category) follows from cocompleteness.

Convolution via exponentiation

In fact, this multicategory has a universal property: viewing $(\mathscr{C}, \otimes, I)$ and $(\mathrm{Set}, \times, I)$ as representable multicategories, the multicategory structure on $\widehat{\mathscr{C}}$ is precisely the exponential

$$\mathrm{Set}^{\mathscr{C}^{op}}$$

in the category of multicategories [Pisani 2014].

Moreover, the exponentiable multicategories are precisely the promonoidal categories [Pisani 2014].

Multicategorical consequences

An advantage of the multicategorical perspective is that it simplifies the study of structures internal to the convolution monoidal structure.

For instance, a **monoid** in a multicategory $\mathcal{M}$ is equivalent to a functor $1 \longrightarrow \mathcal{M}$ of multicategories. Hence, a monoid in $\widehat{\mathcal{C}}$ is a functor $1 \longrightarrow \mathbf{Set}^{\mathcal{C}^{op}}$, thus a functor $\mathcal{C}^{op} \longrightarrow \mathbf{Set}$ of multicategories, i.e. a lax monoidal functor.

The fact that lax monoidal functors are monoids therefore essentially becomes a triviality.

Motivation

From monoidal categories to double categories

Monoidal categories are examples of much richer structures: double categories. While it is often said that monoidal categories are simply one-object bicategories, in practice, it is typically more useful to view monoidal categories as double categories with one object and one tight morphism.

This is because, just as we typically view monoidal categories as categories equipped with structure, so too may we view a double category as structure on a category; conversely, bicategories do not have underlying categories.

Double categories

A **double category** is a category-like structure with two kinds of morphism (**tight** and **loose**), and **cells** filling squares:

$$
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & A' \arrow[l, "p"', "|"{marking}] \arrow[d, "f'"] \\
B & B' \arrow[l, "q", "|"{marking}]
\end{tikzcd}
$$

tight

loose

Cells can be composed in both the tight direction, and the loose direction, but the latter is merely associative and unital up to coherent isomorphism [Grandis & Paré 1999].

Composition in double categories

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A' \arrow[l, "|" marking, "p"'] \arrow[d] & A'' \arrow[l, "|" marking, "p'"'] \arrow[d] \\
B \arrow[ur, phantom, "\varphi"] & B' \arrow[l, "|" marking, "q"] \arrow[ur, phantom, "\varphi'"] & B'' \arrow[l, "|" marking, "q'"]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[d, "f"'] & & A'' \arrow[ll, "|" marking, "p \odot p'"'] \arrow[d] \\
B & \varphi \odot \varphi' & B'' \arrow[ll, "|" marking, "q \odot q'"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A' \arrow[l, "|" marking, "p"'] \arrow[d, "f'"] \\
B \arrow[d, "g"'] \arrow[ur, phantom, "\varphi"] & B' \arrow[l, "|" marking] \arrow[d, "g'"] \\
C \arrow[ur, phantom, "\psi"] & C' \arrow[l, "|" marking, "r"]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[d, "gf"'] & A' \arrow[l, "|" marking, "p"'] \arrow[d, "g'f'"] \\
C \arrow[ur, phantom, "\psi\varphi"] & C' \arrow[l, "|" marking, "r"]
\end{tikzcd}
$$

The underlying categories

A double category $\mathbb{C}$ has two associated categories:

- $\mathbb{C}_0$ is the category whose objects are those of $\mathbb{C}$, and whose morphisms are the tight morphisms.
- $\mathbb{C}_1$ is the category whose objects are the loose morphisms in $\mathbb{C}$ (with arbitrary domain and codomain), and whose morphisms are cells:

$$\begin{tikzcd} A \arrow[d, "f"'] & A' \arrow[l, "p"', "\shortmid"{marking}] \arrow[d, "f'"] \arrow[dl, phantom, "\varphi"] \\ B & B' \arrow[l, "q", "\shortmid"{marking}] \end{tikzcd}$$

$\mathbb{S}\mathrm{pan}$

A prototypical example of a double category is $\mathbb{S}\mathrm{pan}$, whose underlying category of objects and tight morphisms is Set, whose loose morphisms are spans of sets, and whose cells are morphisms of spans.

$$
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & A' \arrow[l, "p"', "\shortmid" marking] \arrow[d, "f'"] \\
B & B' \arrow[l, "q", "\shortmid" marking]
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A \arrow[d, "f"'] & P \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\varphi"] & A' \arrow[d, "f'"] \\
B & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}
$$

$\mathbb{D}\mathrm{ist}$

Another motivating example is $\mathbb{D}\mathrm{ist}$, whose underlying category of objects and tight morphisms is Cat, whose loose morphisms $X \leftarrow\!\!\!\!+\!\!\!- Y$ are distributors, i.e. functors $X^{op} \times Y \longrightarrow \mathrm{Set}$, and whose cells are natural transformations.

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A' \arrow[l, "p"', "\shortmid" marking] \arrow[d, "f'"] \arrow[dl, phantom, "\varphi"] \\
B & B' \arrow[l, "q", "\shortmid" marking]
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
A^{op} \times A' \arrow[r, "p"] \arrow[d, "f^{op} \times f'"'] & \mathrm{Set} \arrow[dl, phantom, "\Swarrow \varphi"] \\
B^{op} \times B' \arrow[ur, bend right, "q"'] &
\end{tikzcd}
$$

Convolution for double categories

It is often illuminating to generalise phenomena from monoidal category theory to double category theory.

Why might we be interested in generalising convolution in particular to double categories?

There are several instances of constructions in the literature that are clearly reminiscent of convolution, but whose relation is mysterious. We should like to see them as instances of a single construction, thereby clarifying the big picture.

Case study 1

Given a bicategory $\mathcal{B}$, we can form a new bicategory $\hat{\mathcal{B}}$, the local cocompletion of $\mathcal{B}$ [Day 1973], whose hom-categories are defined by

$$\hat{\mathcal{B}}(X,Y) := \widehat{\mathcal{B}(X,Y)}$$

and whose composition is defined via a convolution formula:

$$p \mathbin{\hat{\odot}} q := h \mapsto \int^{f \in \mathcal{B}(Y,Z),\, g \in \mathcal{B}(X,Y)} p(f) \times q(g) \times \mathcal{B}(X,Z)(h, f \odot g)$$

Case study 2

Recently, Behr, Melliès & Zeilberger showed that, for each small double category $\mathbb{C}$, the presheaf category $\widehat{\mathbb{C}_1}$ on $\mathbb{C}_1$ (the category of loose morphisms and cells in $\mathbb{C}$) inherits a colax monoidal structure, where the tensor product is defined via a convolution formula [2023]:

$$p \mathbin{\hat{\otimes}} q := h \mapsto \int^{f, g \in \mathbb{C}_2} p(f) \times q(g) \times \mathbb{C}(h, f \odot g)$$

The Yoneda embedding for double categories

We may fruitfully study a category $\mathcal{C}$ by embedding it into the category of presheaves via the Yoneda embedding $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.

If we want to use a similar technique to study a double category $\mathbb{C}$, we are faced with two questions:

- What is a presheaf on a double category?
- What structure do presheaves on a double category form?

Case study 3

Motivated by such questions, Paré showed [2011] that, for any double category $\mathbb{C}$, the contravariant $\mathbb{S}\mathrm{pan}$-valued lax functors therefrom assemble, not into a double category, but a virtual double category

$$\mathbb{L}\mathrm{ax}(\mathbb{C}^{\mathrm{opt}}, \mathbb{S}\mathrm{pan})$$

that plays the role of the presheaf construction for $\mathbb{C}$, in an appropriate sense.

Unification

Virtual double categories

To understand convolution for monoidal categories, we moved to the more general context of multicategories. To understand convolution for double categories, we must move to an analogously more general context.

Just as monoidal categories may be viewed as one-object double categories, multicategories may be viewed as one-object virtual double categories.

Virtual double categories

A **virtual double category** (VDC) is a structure like a double category, except that we drop the assumption that loose morphisms may be composed, and instead permit **multiary cells** [Burroni 1971]:

$$\begin{tikzcd}
A_0 \arrow[d, "f"'] & A_1 \arrow[l, "p_1"'] & \cdots \arrow[l, "p_2"'] & A_n \arrow[l, "p_1"'] \arrow[d, "f'"] \\
B & & & B \arrow[lll, "q"] \arrow[lll, phantom, shift left=6ex, "\varphi"]
\end{tikzcd}$$

Composition in a VDC

Just as in a double category, the tight morphisms assemble into a category. Furthermore, we have identity and composite cells as follows:

$$\begin{tikzcd}
\cdot \arrow[d, equal] \arrow[r, phantom, ""] & \cdot \arrow[l, "p"', "|" marking] \arrow[d, equal] \arrow[dl, phantom, "="] \\
\cdot & \cdot \arrow[l, "p", "|" marking]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdot \arrow[d] & \cdot \arrow[l, "|" marking] & \cdot \arrow[l, "|" marking] \arrow[d] \arrow[dll, phantom, "\varphi_1"] & \cdot \arrow[l, "|" marking] & \cdot \arrow[l, "|" marking] \arrow[d] \arrow[dll, phantom, "\varphi_2"] \\
\cdot \arrow[d] & & \cdot \arrow[ll, "|" marking] & & \cdot \arrow[ll, "|" marking] \arrow[d] \arrow[dllll, phantom, "\psi"] \\
\cdot & & & & \cdot \arrow[llll, "|" marking]
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}[column sep=small]
\cdot \arrow[d] & \cdot \arrow[l, "|" marking] & \cdot \arrow[l, "|" marking] & \cdot \arrow[l, "|" marking] & \cdot \arrow[l, "|" marking] \arrow[d] \arrow[dllll, phantom, "(\varphi_1, \varphi_2) ; \psi"] \\
\cdot & & & & \cdot \arrow[llll, "|" marking]
\end{tikzcd}$$

Representability

Every double category may be viewed as a VDC, in which the multiary cells are the cells whose domain is a composite of loose morphisms. Such VDCs are called representable.

$$\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "p_1"'] & \cdots & \cdot \arrow[l, "p_n"'] \arrow[d] \\
\cdot & & & \cdot \arrow[lll, "q"] \arrow[ulll, phantom, "\varphi"]
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "p_1 \odot \cdots \odot p_n"'] \arrow[d] \\
\cdot & \cdot \arrow[l, "q"] \arrow[ul, phantom, "\varphi"]
\end{tikzcd}$$

Functors between representable VDCs correspond to lax functors between double categories [DPP 2006].

Convolution via exponentiation

Theorem

Every representable VDC (i.e. double category) is exponentiable.

Convolution via exponentiation

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Every representable VDC (i.e. double category) is exponentiable.

Given VDCs $\mathbb{C}$ and $\mathbb{D}$, the underlying category of the exponential $\mathbb{D}^{\mathbb{C}}$ is the functor category $[\mathbb{C}_0, \mathbb{D}_0]$, while a loose morphism is a graph morphism:

$$\begin{array}{ccccc} \mathbb{C}_0 & \leftarrow & \mathbb{C}_1 & \rightarrow & \mathbb{C}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}_0 & \leftarrow & \mathbb{D}_1 & \rightarrow & \mathbb{D}_0 \end{array}$$

Case study 1

Every bicategory $\mathcal{B}$ may be viewed as a double category with no nontrivial tight morphisms. In particular, bicategories are exponentiable VDCs.

Proposition

The full sub-VDC of $\mathbb{S}\mathrm{pan}^{\mathcal{B}^{\mathrm{co}}}$ spanned by the objects of $\mathcal{B}$, i.e. the representable presheaves, is again a bicategory, and exhibits the local cocompletion of $\mathcal{B}$.

Case study 2

Denote by $\Sigma\mathcal{M}$ the one-object VDC associated to a multicategory $\mathcal{M}$.

Proposition

Given a double category $\mathbb{C}$, $(\Sigma\mathbf{Set})^{\mathbb{C}^{\mathrm{opt}}}$ is a weakly representable multicategory, whose underlying category is $\widehat{\mathbb{C}}_1$ and which exhibits the colax monoidal structure of [BMZ 2023].

The VDC of monads

Recall that a monoid in the convolution monoidal structure on $\widehat{\mathscr{C}}$ is precisely a lax monoidal functor from $\mathscr{C}^{op}$ to $\mathbf{Set}$.

A monoid is to a multicategory what a monad is to a VDC.

$$
\begin{tikzcd}
\bullet \arrow[d, equal] & \bullet \arrow[l, "T"', "|" marking] \arrow[d, phantom, "\mu"] & \bullet \arrow[l, "T"', "|" marking] \arrow[d, equal] \\
\bullet & & \bullet \arrow[ll, "T", "|" marking]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, equal] \arrow[r, equal] \arrow[dr, phantom, "\eta"] & \bullet \arrow[d, equal] \\
\bullet & \bullet \arrow[l, "T", "|" marking]
\end{tikzcd}
$$

For every VDC $\mathbb{X}$, there is a VDC $\mathbf{Mod}(\mathbb{X})$ whose objects are monads in $\mathbb{X}$, and whose loose morphisms are bimodules [Leinster 2004].

Case study 3

The objects of the exponential VDC $\mathbb{D}^{\mathbb{C}}$ are functors $\mathbb{C}_0 \longrightarrow \mathbb{D}_0$, rather than lax functors. However, we can obtain the lax functors using the intuition on the the previous slide.

Theorem

Given double categories $\mathbb{C}$ and $\mathbb{D}$, Paré's VDC $\mathbb{L}\mathrm{ax}(\mathbb{C},\mathbb{D})$ of lax functors from $\mathbb{C}$ to $\mathbb{D}$ is precisely $\mathbb{M}\mathrm{od}(\mathbb{D}^{\mathbb{C}})$.

Yoneda embedding for double categories

Paré's motivation for introducing $\mathbb{L}\mathrm{ax}(\mathbb{C}, \mathbb{D})$ is to study the Yoneda lemma for double categories. Paré explicitly constructs an embedding

$$\mathbb{C} \longrightarrow \mathbb{L}\mathrm{ax}(\mathbb{C}^{\mathrm{opt}}, \mathbb{S}\mathrm{pan})$$

which takes quite some work, as the VDC of presheaves is rather complex.

However, our characterisation in terms of exponentials simplifies this construction substantially.

Yoneda via transposition

normal functors $\mathbb{C} \longrightarrow \mathbb{L}\mathrm{ax}(\mathbb{C}^{\mathrm{opt}}, \mathbb{S}\mathrm{pan})$

normal functors $\mathbb{C} \longrightarrow \mathbb{M}\mathrm{od}(\mathbb{S}\mathrm{pan}^{\mathbb{C}^{\mathrm{opt}}})$

functors $\mathbb{C} \longrightarrow \mathbb{S}\mathrm{pan}^{\mathbb{C}^{\mathrm{opt}}}$

functors $\mathbb{C}^{\mathrm{opt}} \times \mathbb{C} \longrightarrow \mathbb{S}\mathrm{pan}$

Indexation versus fibration

In category theory, one finds many correspondences between indexed data and fibred data.

- $\mathbf{Set}^X$ versus $\mathbf{Set}/X$.
- $[\mathscr{C}^{op}, \mathbf{Set}]$ versus $\mathbf{DFib}(\mathscr{C})$.
- $\mathbf{Ps}(\mathscr{C}^{op}, \mathbf{Cat})$ versus $\mathbf{Fib}(\mathscr{C})$.

It is known that something similar is true for double categories [Paré 2011, Lambert 2021, CLPS 2022, FM24], albeit under certain strictness assumptions.

Presheaves and discrete fibrations

By taking advantage of the observation that lax functors are monads, we can give a far simpler proof, whilst dropping unnecessary strictness assumptions.

First, the usual equivalence between presheaves and discrete fibrations induces an equivalence of VDCs:

$$\mathbb{S}\mathrm{pan}^{\mathbb{C}^{\mathrm{op}}} \simeq \mathbb{L}\mathrm{vwDFib}(\mathbb{C})$$

Presheaves and discrete fibrations

After which applying $\mathbb{M}\mathrm{od}$ and invoking its universal property gives an equivalence of VDCs:

$$\mathbb{L}\mathrm{ax}(\mathbb{C}^{\mathrm{opt}}, \mathbb{S}\mathrm{pan}) \simeq \mathbb{D}\mathrm{Fib}(\mathbb{C})$$

for every double category $\mathbb{C}$.

Furthermore, using the universal property of $\mathbb{M}\mathrm{od}$:

$$\mathbb{L}\mathrm{ax}(\mathbb{C}^{\mathrm{opt}}, \mathbb{S}\mathrm{pan}) \simeq \mathbb{L}\mathrm{ax}_N(\mathbb{C}^{\mathrm{opt}}, \mathbb{D}\mathrm{ist})$$

normal functors

Epilogue

Read all about it

If you'd like to learn more, you can find (almost) everything I've talked about, and much more, in the preprint:

'Exponentiable virtual double categories
and
presheaves for double categories'

https://arxiv.org/abs/2508.11611

Summary

- Convolution is simplified and clarified by the consideration of multicategories.
- Taking a similar perspective for convolution with 'many objects' leads to virtual double categories.
- By studying exponentiation for virtual double categories, we recover many known phenomena, whilst simplifying and generalising existing proofs.

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Postscript

More generally than considering monoids in $\hat{\mathscr{C}}$, we can consider categories enriched therein. Again, the multicategorical perspective simplifies the analysis, revealing that $\hat{\mathscr{C}}$-categories are precisely $\mathscr{C}$-graded categories.

In fact, these are precisely the monads in $\mathbf{Span}^{\Sigma\mathscr{C}^{op}}$.

Consequently, monads in $\mathbf{Span}^{\mathbb{C}^{opt}}$ may be viewed as categories graded in a double category...